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The quantum channel subject to local interaction with two-level environment is studied. The two-level environment is regarded as a quantum bit (qubit) as well as a pair of particles owned by Alice and Bob. The amount of entanglement initially shared by Alice and Bob is distributed among these three qubits due to the interaction. In this model, we show that the singlet fraction of the decohered quantum channel is uniquely determined by the distributed entanglement. When the decohered quantum channel is used under the standard teleportation scheme, the optimal teleportation fidelity is well understood by considering the remaining entanglement between environment and transmitted state.

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Entanglement is an important resource for most applications of quantum information, and a number of measures quantifying the amount of entanglement, such as the entanglement of formation [1], have been proposed. In the quantum teleportation [2], two bits of classical information and a pair of particles in a maximally entangled state can transmit an unknown quantum state faithfully. The entangled particles owned by Alice and Bob act as a quantum channel carrying quantum information. In the *standard teleportation scheme* (STS) [2,3], Horodecki showed that the optimal fidelity of the teleported state is given by [4]

$$f = \frac{2F_{AB} + 1}{3}, \quad (1)$$

where F_{AB} is the singlet fraction of the channel state ϱ_{AB} in the bipartite 2×2 system. The singlet fraction is defined as $F_{AB} = \max\langle e|\varrho_{AB}|e\rangle$, where the maximum is taken over the all maximally entangled states $|e\rangle$ [1].

When the quantum channel is a pure state, the singlet fraction can be regarded as a measure of entanglement. In fact, the singlet fraction is related to the Hill-Wootters concurrence [5,6] as $F_{AB} = (1 + C_{AB})/2$. Therefore, the optimal fidelity in the STS is written by the concurrence as

$$f = \frac{2}{3} + \frac{1}{3}C_{AB}. \quad (2)$$

However, when the quantum channel is a mixed state, the singlet fraction is no longer the measure of entanglement. The entanglement does not increase under the local quantum operations and classical communications,

but the singlet fraction does. In fact, recently Badziag, Horodecki [3], and Bandyopadhyay [7] showed this case.

When the quantum channel interacts with the surrounding environment, the quantum channel is entangled with the environment and falls into the mixed state. When such the decohered quantum channel is used for the quantum teleportation, the teleported state is generally entangled with the environment, and some relation between the remaining entanglement and teleportation fidelity will be expected.

In this report, we shall investigate the quantum channel subject to local interaction with environment. We restrict ourselves to the case that the environment is a two-level system for simplicity. Therefore, the environment is regarded as a quantum bit (qubit) as well as a pair of particles owned by Alice and Bob, and three qubits constitute the total system. In this model, the amount of entanglement initially shared by Alice and Bob is distributed among these three qubits due to the interaction. We show that the singlet fraction of the decohered quantum channel is uniquely determined by the distributed entanglement. When the decohered quantum channel is used under the STS, the optimal teleportation fidelity is well understood by considering the remaining entanglement between transmitted state and environment.

Let Alice and Bob initially share an ideal (perfect) quantum channel: a pair of particles in the maximally entangled state of $|\phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. Hereafter, each qubit of the quantum channel is denoted by A and B , respectively, and the qubit of the environment is denoted by E . When the initial state of the environment is denoted by $|0\rangle_E$, the initial total state is

$$\varrho_{ABE} = P_{AB}^+ \otimes |0\rangle_{EE}\langle 0|, \quad (3)$$

where $P_{AB}^+ = |\phi^+\rangle\langle\phi^+|$. Then, we assume that only one particle of the channel (say Alice's qubit) is subject to the interaction with the environment. Any type of the interaction is described by the $SU(4)$ unitary matrix acting on Alice's qubit and environment. Therefore, the total state after the interaction is given by

$$\varrho'_{ABE} = U_{AE}(P_{AB}^+ \otimes |0\rangle_{EE}\langle 0|)U_{AE}^\dagger. \quad (4)$$

The quantum channel is decohered due to the interaction, and the reduced density matrix of the pair of AB is written in the Stinespring form [8] as

$$\varrho'_{AB} = \text{Tr}_E \varrho'_{ABE} = \sum_k (M_k \otimes 1) P_{AB}^+ (M_k^\dagger \otimes 1)$$

$$\equiv (\Lambda \otimes 1)P_{AB}^+, \quad (5)$$

where

$$M_k = {}_E \langle k | U_{AE} | 0 \rangle_E, \quad (6)$$

and $\sum_k M_k^\dagger M_k = 1$. In the same manner, the reduced density matrix of the pair of BE is written as

$$\begin{aligned} \varrho'_{EB} &= \text{Tr}_A \varrho'_{ABE} = \sum_k (N_k \otimes 1) P_{AB}^+ (N_k^\dagger \otimes 1) \\ &\equiv (\Gamma \otimes 1) P_{AB}^+, \end{aligned} \quad (7)$$

where

$$N_k = {}_A \langle k | U_{AE} | 0 \rangle_E, \quad (8)$$

and $\sum_k N_k^\dagger N_k = 1$.

It should be noted here that the above model has two considerable properties. First is related to the recent work by Coffman, Kundu, and Wootters [9], where a relation in distributing entanglement among three qubits was shown. Since ϱ'_{ABE} is a pure state, applying the relation directly to our model, we obtain

$$C_{AB}^2 + C_{EB}^2 + \tau_{ABE} = C_{B(AE)}^2 = 1, \quad (9)$$

where C_{AB} and C_{EB} is the concurrence between A and E , and between B and E , respectively. τ_{ABE} is the three-qubit entanglement of the triple of ABE , and $C_{B(AE)}$ is the concurrence between B and the pair of AE . Since U_{AE} acts only on the pair of AE , the transformation preserves $C_{B(AE)}$, which is equal to the initial entanglement between A and B , that is unity. Therefore, in our model, U_{AE} plays a role to distribute the entanglement initially shared by Alice and Bob to two-qubit entanglement (AB and BE) and three-qubit entanglement.

Second is a special relation established between ϱ'_{AB} and ϱ'_{EB} . Choi [10] and Horodecki family [4] showed an isomorphism between completely positive trace-preserving (CPTP) maps and quantum states with one of subsystems being completely mixed. Since both ϱ'_{AB} and ϱ'_{EB} is obtained by CPTP map from P_{AB}^+ , the map Λ and the state ϱ'_{AB} is isomorphic, and Γ and ϱ'_{EB} is also isomorphic. When some ϱ'_{AB} is given, the map Λ is uniquely determined due to the isomorphism. Although the Stinespring form of the given Λ is not unique, any two Stinespring forms of the same Λ are related as $M_k^{(1)} = \sum_\mu U_{k\mu} M_\mu^{(2)}$ with U being unitary. Further, from Eq. (6) and Eq. (8), the matrix elements of M and N are related to each other through $[N_k]_{ij} = [M_i]_{kj}$. Therefore, two Stinespring forms of Γ determined by $M_k^{(1)}$ and $M_k^{(2)}$ are related as $N_k^{(1)} = U N_k^{(2)}$, where U is local unitary acting on E . As a result, for some given ϱ'_{AB} , all the corresponding ϱ'_{EB} 's are in a locally equivalent class.

Figure 1 shows the singlet fraction (F_{AB}) and the concurrence (C_{AB}) of ϱ'_{AB} , which are obtained numerically for 10 000 random U_{AE} 's in the circular unitary ensemble [11]. Since ϱ'_{AB} is mixed states, the points broadly

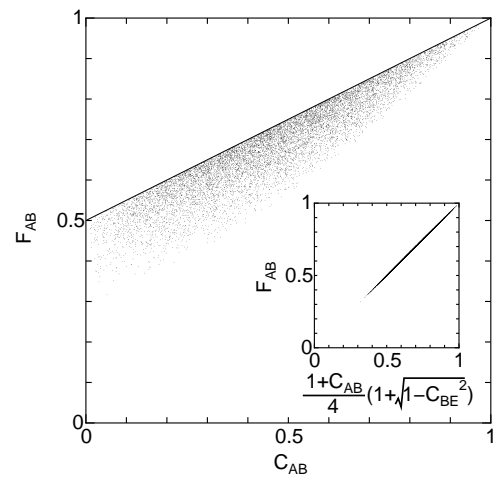


FIG. 1. Numerically obtained singlet fraction (F_{AB}) and concurrence (C_{AB}) in our model (both are dimensionless). A solid line shows $F_{AB} = (C_{AB} + 1)/2$. The inset shows F_{AB} as a function of $(1 + C_{AB})(1 + \sqrt{1 - C_{BE}^2})/4$.

distribute in the region of $F_{AB} \leq (C_{AB} + 1)/2$, and we cannot see any unique relation between F_{AB} and C_{AB} . However, since all ϱ'_{EB} 's corresponding to ϱ'_{AB} are in a locally equivalent class as mentioned above, it is expected that some nonlocal properties of ϱ'_{AB} is transferred to ϱ'_{EB} , and F_{AB} might be uniquely determined by C_{AB} with the help of some nonlocal property of ϱ'_{EB} , as it will be shown below.

In order to see this, we first examine an example for a simple case of U_{AE} as follows:

$$\begin{cases} |00\rangle_{AE} \longrightarrow \sqrt{1-q}|00\rangle_{AE} + \sqrt{q}|11\rangle_{AE} \\ |10\rangle_{AE} \longrightarrow \sqrt{1-p}|10\rangle_{AE} + \sqrt{p}|01\rangle_{AE}. \end{cases} \quad (10)$$

When $q=0$, this transformation corresponds to the usual amplitude damping for Alice's qubit. The reduced density matrix of ϱ'_{AB} are easily obtained as

$$\begin{aligned} \varrho'_{AB} &= \frac{1-q}{2}|00\rangle\langle 00| + \frac{p}{2}|01\rangle\langle 01| + \frac{q}{2}|10\rangle\langle 10| \\ &+ \frac{1-p}{2}|11\rangle\langle 11| + \left[\frac{\sqrt{(1-p)(1-q)}}{2}|00\rangle\langle 11| \right. \\ &\left. + \frac{\sqrt{pq}}{2}|01\rangle\langle 10| + \text{h.c.} \right], \end{aligned} \quad (11)$$

and the singlet fraction and concurrence of ϱ'_{AB} is

$$\begin{aligned} C_{AB} &= |\sqrt{(1-p)(1-q)} - \sqrt{pq}|, \\ F_{AB} &= \frac{2 - p - q + 2\sqrt{(1-p)(1-q)}}{4}. \end{aligned} \quad (12)$$

Further, we obtain

$$\begin{aligned} \varrho'_{EB} &= \frac{1-q}{2}|00\rangle\langle 00| + \frac{1-p}{2}|01\rangle\langle 01| + \frac{q}{2}|10\rangle\langle 10| \\ &+ \frac{p}{2}|11\rangle\langle 11| + \left[\frac{\sqrt{p(1-q)}}{2}|00\rangle\langle 11| \right. \end{aligned}$$

$$+ \frac{\sqrt{(1-p)q}}{2} |01\rangle\langle 10| + \text{h.c.} \Big], \quad (13)$$

whose concurrence is

$$C_{EB} = |\sqrt{(1-p)q} - \sqrt{p(1-q)}|. \quad (14)$$

Combining Eq. (12) and Eq. (14), we finally obtain a rather simple relation of

$$F_{AB} = \frac{1 + C_{AB}}{4} \left(1 + \sqrt{1 - C_{EB}^2} \right). \quad (15)$$

Although the above relation was obtained for a special case of U_{AE} , we found numerically that the relation holds for any form of U_{AE} . In fact, F_{AB} numerically obtained for random U_{AE} 's linearly depends on $(1 + C_{AB})(1 + \sqrt{1 - C_{EB}^2})/4$ as shown in the inset of Fig. 1. Since all quantities in Eq. (15) are invariant under any local unitary, Eq. (15) holds for any maximally entangled state as an initial state of AB . It is nontrivial that the above simple relation holds, since U_{AE} has $15 - 2 \times 3 = 9$ independent nonlocal parameters. Then, we arrive at the main result of this report: *When an ideal quantum channel (any maximally entangled state) is locally decohered due to any interaction with two-level environment (another qubit), the singlet fraction of the decohered quantum channel is determined uniquely by the distributed entanglement as Eq. (15).*

It should be noted here that, when the initial state of AB is partially entangled pure state, we could not find any unique relation. This may be because the special relation between ϱ'_{AB} and ϱ'_{EB} , which is not established for partially entangled initial states, plays a crucial role.

According to Eq. (1) by Horodecki [4], when the decohered quantum channel ϱ'_{AB} is used under the STS, the optimal fidelity is given by

$$f = \frac{1}{2} + \frac{\sqrt{1 - C_{EB}^2}}{6} + \frac{1 + \sqrt{1 - C_{EB}^2}}{6} C_{AB}. \quad (16)$$

Let's consider the physical meaning of f in the STS instead of F_{AB} itself.

The unknown state to be teleported is assumed to be pure state of $\varrho_u = (1 + \vec{s} \cdot \vec{\sigma})/2$ for simplicity ($|\vec{s}|=1$). In the STS, Alice performs a collective measurement for Alice's qubit and an unknown state (see Fig. 2 (a)). Alice obtains the result of the measurement α with a probability P_α . After the measurement, the state of the pair of EB becomes a pure state $|\psi_\alpha\rangle$, which depends on α . This represents a decomposition of the mixed state ϱ'_{EB} into pure states:

$$\varrho'_{EB} = \sum_{\alpha} P_{\alpha} |\psi_{\alpha}\rangle\langle\psi_{\alpha}|. \quad (17)$$

Bob's qubit is still entangled with the environment, but the amount of the entanglement also depends on α .

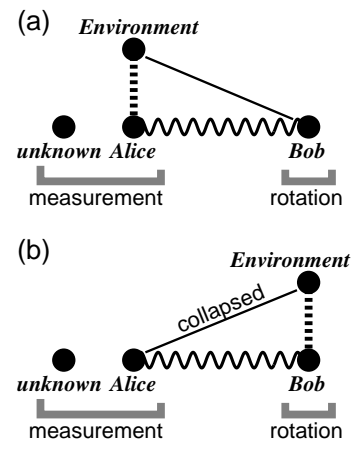


FIG. 2. When the decohered quantum channel in our model is used under the STS, two types of configuration are present. (a) Alice's qubit only and (b) Bob's qubit only is coupled with the environment.

When the concurrence of $|\psi_\alpha\rangle$ is denoted by C_{EB}^α , convexity of the concurrence implies

$$C_{EB} \leq \sum_{\alpha} P_{\alpha} C_{EB}^{\alpha}. \quad (18)$$

Then, Bob rotates the state of his qubit depending on α , and $|\psi_\alpha\rangle$ becomes $|\phi_\alpha\rangle$. However, since the rotation is local in the STS, C_{EB}^α does not change. Finally, Bob obtains the state of $\varrho_B^\alpha = \text{Tr}_E |\phi_\alpha\rangle\langle\phi_\alpha|$, whose eigenvalues are

$$\left\{ \frac{1}{2} + \frac{q_\alpha}{2}, \frac{1}{2} - \frac{q_\alpha}{2} \right\}. \quad (19)$$

Since $|\phi_\alpha\rangle$ is a pure state, the eigenvalues of ϱ_B^α are related to C_{EB}^α as

$$q_\alpha = \sqrt{1 - C_{EB}^{\alpha 2}}. \quad (20)$$

When the quantum channel is absent and only classical communications are allowed, it will be optimal that Alice and Bob adopt the following strategy: Alice performs an orthogonal measurement in a spin direction (say z -axis) on the unknown state and Bob prepares a state depending on the result of Alice's measurement. Alice measures the spin "up" and "down" with probability $p_\uparrow = (1 + s_z)/2$ and $p_\downarrow = (1 - s_z)/2$, respectively. In this strategy, if Bob's particle is in a pure state, Bob can prepare the state

$$\varrho_B = \frac{p_\uparrow}{2} [1 + \sigma_z] + \frac{p_\downarrow}{2} [1 - \sigma_z] = \frac{1}{2} [1 + s_z \sigma_z]. \quad (21)$$

However, since Bob's particle must be entangled with the environment, Bob's particle must be in the mixed state with eigenvalues of Eq. (19) for each result of Alice's measurement "up" and "down". Under this constraint, Bob can only prepare

$$\varrho_B^\alpha = \frac{p_\uparrow}{2} [1 + q_\alpha \sigma_z] + \frac{p_\downarrow}{2} [1 - q_\alpha \sigma_z] = \frac{1}{2} [1 + q_\alpha s_z \sigma_z]. \quad (22)$$

Therefore, using classical communications only, the attainable fidelity averaged over \vec{s} and α is

$$\begin{aligned} f^{\text{CC}} &= \sum_{\alpha} P_{\alpha} \left(\frac{1}{2} + \frac{q_{\alpha}}{6} \right) = \sum_{\alpha} P_{\alpha} \left(\frac{1}{2} + \frac{\sqrt{1 - C_{BE}^2}}{6} \right) \\ &\leq \frac{1}{2} + \frac{1}{6} \sqrt{1 - \left(\sum_{\alpha} P_{\alpha} C_{BE}^2 \right)} \\ &\leq \frac{1}{2} + \frac{\sqrt{1 - C_{BE}^2}}{6} \equiv f_{\text{max}}^{\text{CC}} \end{aligned} \quad (23)$$

Here, we have used the concavity of $\sqrt{1 - x^2}$ for the first inequality and convexity of the concurrence Eq. (18) for the second inequality. The upper bound of $f_{\text{max}}^{\text{CC}}$ agrees with Eq. (16) for $C_{AB} = 0$.

When an ideal quantum channel is shared by Alice and Bob and if Bob's particle is in a pure state, Bob can prepare ϱ_u faithfully. However, Bob's particle must be in the mixed state as discussed above. The optimal state under the constraint is thus

$$\varrho_B^{\alpha} = \frac{1}{2} [1 + q_{\alpha} \vec{s} \cdot \vec{\sigma}]. \quad (24)$$

The fidelity averaged over \vec{s} and α is again

$$f^{\text{QC}} = \sum_{\alpha} P_{\alpha} \left(\frac{1}{2} + \frac{q_{\alpha}}{2} \right) \leq \frac{1}{2} + \frac{\sqrt{1 - C_{BE}^2}}{2} \equiv f_{\text{max}}^{\text{QC}}. \quad (25)$$

As a result, the fidelity in our model Eq. (16) is rewritten as

$$f = f_{\text{max}}^{\text{CC}} + \frac{f_{\text{max}}^{\text{QC}}}{3} C_{AB}. \quad (26)$$

The meaning of the factor $1/3$ is that a half of the fidelity is gained by the maximally mixed state, and a remaining half is gained by the preparation, but one of three degrees of \vec{s} is already used in $f_{\text{max}}^{\text{CC}}$. Therefore, $1/2 \times 2/3 = 1/3$ of $f_{\text{max}}^{\text{QC}}$ is carried by the entanglement between Alice and Bob, which linearly depends on the concurrence C_{AB} . The above expression Eq. (26) can be regarded as the natural extension of Eq. (2) considering the remaining entanglement between teleported state and the environment. In fact, Eq. (26) completely agrees with Eq. (2) for $C_{EB} = 0$. It is interesting to note that the fidelity is determined by the upper bound of f^{CC} and f^{QC} . In this sense, the STS seems to be optimal under the constraint of the remaining entanglement, at least in our model.

In the above, we consider the case that Alice's qubit is coupled with the environment. When Bob's qubit is coupled with the environment as shown in Fig. 2 (b), C_{EB} in Eq. (15) must be read as C_{EA} . In this case also, Bob's qubit is entangled with the environment after the procedure of STS. However, the state of the pair of EB has no special relation to the state of the quantum channel. The state ϱ'_{EA} , which has a special relation, is collapsed by Alice's measurement. Therefore, it is difficult to discuss the physical meaning when Bob's qubit is coupled with

the environment. Further, in this configuration, only an inequality

$$\begin{aligned} f &= \frac{3 + \sqrt{1 - C_{AE}^2}}{6} + \frac{1 + \sqrt{1 - C_{AE}^2}}{6} C_{AB} \\ &\leq \frac{3 + \sqrt{1 - C_{EB}^2}}{6} + \frac{1 + \sqrt{1 - C_{EB}^2}}{6} C_{AB} \end{aligned} \quad (27)$$

is satisfied, since we numerically confirmed that $C_{AE} \geq C_{BE}$ in our model, though results are not shown explicitly here. In this sense, the STS for the configuration of Fig. 2 (b) might be less optimal than that of Fig. 2 (a) under the constraint of the remaining entanglement.

It is important to note that the teleportation fidelity itself is the same in both configurations of Fig. 2 (a) and (b), since F_{AB} is the same independent of the configuration. In this sense, the expression of F_{AB} should be symmetric under the exchange of A and B . For this purpose, Eq. (15) can be rewritten by using Eq. (9) as

$$F_{AB} = \frac{1 + C_{AB}}{4} \left(1 + \sqrt{C_{AB}^2 + \tau_{ABE}} \right), \quad (28)$$

which is symmetric since τ_{ABE} is symmetric [9]. However, since three-qubit entanglement τ_{ABE} is also collapsed by the Alice's measurement, it will be hard to discuss the physical meaning in the STS.

In this report, we exclusively consider the case that the environment is a two-level system. In order to discuss general local decoherence of the quantum channel, at least four-level environment must be considered. Even in this case, the special relation between states with respect to the nonlocal properties, which plays a crucial role in this report, is also established. However, the measure of the entanglement for the 2×4 system is necessary for this purpose, and it is important to clarify the nature of entanglement in such larger dimensional systems.

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